Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 1, pp. 29–36, 2002

A NEW APPROACH TO STUDYING THE DYNAMICS OF A THIN CURVED VORTEX

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UDC 532.527

The paper studies the dynamics of a thin curved vortex in a potential flow of an ideal incompressible fluid. The flow is specified by a number of geometrical restrictions and does not satisfy the Biot–Savart law. The form of the derived equation of the vortex dynamics coincides with the form of the well-known equation of local induction for self-induced vortex motion. The parameters of the new equation are simultaneously flow parameters, and in this sense, they do not show uncertainty typical of classical equations. The coefficient of the new equation can take any specified values (not necessarily much greater than unity, as required according to the concept of local induction) and generally is a function of a natural filament parameter.

Introduction. The well-known analytical methods for determining the dynamics of a thin, optionally weakly-curved vortex submerged in a potential flow of an ideal incompressible fluid are based on the assumption that this flow is induced by the vortex, at least at sufficiently large distances from the vortex. It is assumed that the interaction pattern is defined by the Biot–Savart law, which relates the spatial vorticity distribution to the induced velocity.

The simplest approach (see, for example, [1]) considers the vortex as a filament, i.e., the fluid flow inside and near the vortex core is ignored. The vortex dynamics is determined directly from the Biot–Savart integral, and the known singularities in the integral are eliminated using two assumptions: 1) local induction approximation, i.e., the assumption that a neighboring segment of finite length l much larger than the vortex core radius makes a dominant contribution to the velocity of a filament's particle; 2) the assumption that a neighboring segment of length ε of the order of the vortex core radius makes a negligibly small contribution to the particle velocity. In a sense, the second assumption is a consequence of the first assumption. The so-called equation of local induction obtained using the indicated approach can be written as

$$\frac{\partial \boldsymbol{X}}{\partial t} = \frac{\Gamma}{8\pi} \ln\Big(\frac{l}{\varepsilon}\Big) i\psi^* \boldsymbol{N} + \text{c.c.}, \tag{1}$$

where $\boldsymbol{X}(t, s)$ is the radius-vector of a filament's particle as a function of time t and the natural parameter (arc length) of the filament s, Γ is the filament circulation, i is imaginary unit, ψ is the natural curvature of the filament, and $\boldsymbol{N}$ is the normal component of the natural reference point of the filament; the asterisk and c.c. denote complex conjugation.

We note that Eq. (1) belongs to the class of equations integrated using the inverse-scattering method and is equivalent to well-known equations such as the Heisenberg magnetic equation and the nonlinear Schrödinger equation [2].

In papers [3–7], a more systematic analysis of flow is given, and the fluid flow inside and near the vortex core is taken into account. The vortex dynamics is determined by joining asymptotic expansions in ε of the internal and external fields of fluid velocities, and the external field is determined from the Biot–Savart integral in the local induction approximation. The equation obtained in the first approximation in ε differs from Eq. (1) by the presence

Pacific Oceanologic Institute, Far East Division, Russian Academy of Sciences, Vladivostok 690041. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 1, pp. 36–44, January–February, 2002. Original article submitted August 16, 2000; revision submitted July 9, 2001.

0021-8944/02/4301-0029 $27.00

of an additional component in the factor at $i\psi^* \boldsymbol{N}$, whose value depends on the profile of the initial approximation to the axial and azimuthal fluid velocities in the vortex core.

Leibovich, et al. [8, 9] describe various methods for studying the dynamics of a weakly curved vortex, i.e., a vortex whose characteristic deviation from a straight vortex does not exceed the vortex core radius. In the present paper, we do not consider the dynamics of this type of vortices.

The concept of local induction underlying the approaches indicated above cannot be considered adequate. First, within the framework of this concept, the parameters l and ε remain undefined. Second, the local induction approximation restricts the possible values of the equation coefficient by the condition $\ln(l/\varepsilon) \gg 1$. Third, the requirement of constant length l along the filament cannot apparently be satisfied for a vortex of an arbitrary shape. We note that in the case of a weakly curved vortex, the concept of local induction does not agree with experimental data. For example, in [10], the factor $\ln(l/\varepsilon)$ was estimated within unity, and in [11], this factor is a function of local vortex curvature.

In the present paper, we propose to derive the equation of dynamics for a thin curved vortex for the potential flow that does not satisfy the Biot–Savart law. As in [3–7], we obtained the equation of vortex dynamics by joining asymptotic expansions of the internal and external fields of fluid velocities. However, unlike in the papers cited, in which the external field is determined from the Biot–Savart integral in the local induction approximation, we imposed a number of natural constraints on the flow that do not contradict hydrodynamic equations. Under these constraints, the vortex dynamics is described by an equation whose form coincides with that of the equation of local induction. The parameters of the new equation, which are simultaneously flow parameters, are no longer uncertain. The coefficient of the new equation can be an arbitrary function of the natural parameter of the filament. This function specifies both the vortex dynamics and the external flow in which this dynamics occurs.

Generally, the indicated constraints reduce to the requirement that in a small neighborhood of an arbitrary filamentary particle, the external field should coincide with the field induced by a straight filament that is a tangent to the initial vortex filament at this point. This requirement can be considered a new principle for determining the dynamics of a thin curved vortex and a reasonable alternative to the local induction concept.

The external field can be determined by construction of a special coordinate system for an arbitrary vortex filament in a small neighborhood. In this system, one of the coordinates is identified with the external-field potential and the other two coordinates parametrize an arbitrary equipotential field surface. Obviously, any restriction imposed on the external flow is simultaneously a restriction on the coordinates of this system. The system is constructed by expansion in a small parameter and is used as a working system in determining the internal field from hydrodynamic equations.

1. Preliminary Data. We assume that a world volume of fluid is immersed in the coordinate Galilean space $\mathbb{R} \times \mathbb{R}^3$. This assumption makes values dimensionless and distinguishes a characteristic scale — unity. To this scale unit, we will relate the time parameter of vortex motion t, the natural parameter of the vortex s, the characteristic radius of vortex curvature, and vortex circulation Γ. The quantity $\varepsilon \ll 1$ characterizes distances of the same order as the vortex core radius.

Let U_t and γ_t denote the fluid region and vortex filament in $\mathbb{R}^3$, respectively, at time t. We specify the natural curvature ψ and the reference $\langle \boldsymbol{N}, \boldsymbol{t} \rangle$ of the filament γ_t by the equations

$$\psi = æ \exp\left(i \int^s \tau \, ds'\right), \quad \boldsymbol{N} = (\boldsymbol{n} + i\boldsymbol{b}) \exp\left(i \int^s \tau \, ds'\right), \quad \boldsymbol{t} = \boldsymbol{X}_s,$$

where æ is the curvature, τ is the torsion, $\boldsymbol{n}$ and $\boldsymbol{b}$ are the normal and binormal to the filament, respectively, and $\boldsymbol{X}(t,s)$ is the natural parametrization of the filament γ_t. Then, the Serret–Frenet formulas are valid:

$$\boldsymbol{N}_s = -\psi \boldsymbol{t}, \qquad \boldsymbol{t}_s = (\psi^* \boldsymbol{N} + \psi \boldsymbol{N}^*)/2. \tag{2}$$

As the vortex filament, we take a curve in $\mathbb{R}^3$ together with a numerical parameter that means circulation of this filament. A classical expression for the field induced by a vortex filament is given by the Biot–Savart integral.

In the formulas, unless otherwise specified, all subscripts and superscripts, except for n, vary from 1 to 3. The subscript n is an arbitrary integer. Summation is performed over repeated indices.

2. Formulation of the Problem of Determining the External Field. The restrictions imposed on the external potential flow can be considered as requirements that determine the coordinate system (r, φ, s) at an arbitrary time t, where φ is the external-field potential at time t and r and s are some coordinates on the equipotential surface φ. The coordinate system (r, φ, s) is defined in a certain region of the Euclidean space $\mathbb{R}^3$; therefore, it is possible to describe this system in terms of components of the metric induced by a scalar product in $\mathbb{R}^3$. Thus, the problem considered reduces to determining the metrics components in the coordinate system (r, φ, s) at an arbitrary time t.

At any time t, the coordinate system (r, φ, s) is associated with the filament γ_t. As the domain of definition of the coordinates, we take a neighborhood $U_t \setminus \gamma_t$ of the filament γ_t located rather close to γ_t, i.e., the distance from any point of the neighborhood U_t to γ_t is of the same asymptotic order as ε. We require that in this neighborhood as $r \to 0$, the coordinates r, φ, and s form a local (at any point of the filament γ_t) cylindrical coordinate system with the natural parameter s along the filament. In particular, this requirement means that in a small neighborhood of an arbitrary point of the filament γ_t, the external field coincides with the field induced by a straight vortex filament tangent to γ_t at this point.

We show how to define the coordinates r, φ, and s by the use of metrics. Here we are dealing with the vector-field basis $(\partial_1, \partial_2, \partial_3)$ in $U_t \setminus \gamma_t$, which is related to the coordinate basis $(\partial_r, \partial_\varphi, \partial_s)$ of the system (r, φ, s) by

$$\partial_1 = \partial_r, \qquad \partial_2 = r^{-1}\partial_\varphi, \qquad \partial_3 = \partial_s.$$

We use $\boldsymbol{x} = \boldsymbol{x}(t, r, \varphi, s)$ to denote the mapping of conversion from the sought coordinates in $U_t \setminus \gamma_t$ to Cartesian coordinates and set

$$\boldsymbol{e}_k := \partial_k \boldsymbol{x}. \tag{3}$$

Using the vector fields (3), we determine h_{kl}, b^m_{kl}, and γ^m_{kl}, assuming that

$$h_{kl} = (\boldsymbol{e}_k, \boldsymbol{e}_l), \qquad \partial_l \boldsymbol{e}_k - \partial_k \boldsymbol{e}_l = b^m_{kl}\boldsymbol{e}_m, \qquad \partial_l \boldsymbol{e}_k = \gamma^m_{kl}\boldsymbol{e}_m, \tag{4}$$

where $(\cdot, \cdot)$ is a scalar product in $\mathbb{R}^3$. The quantities h_{kl} are the metrics components in $U_t \setminus \gamma_t$ in the basis $(\partial_1, \partial_2, \partial_3)$. The coefficients b^m_{kl} are calculated from the formula

$$b^m_{kl} = r^{-1}\delta^m_2(\delta^1_k\delta^2_l - \delta^1_l\delta^2_k), \tag{5}$$

where δ^k_l is the Kronecker symbol. The quantities γ^m_{kl} are called the coefficients of canonical plane connectivity in $U_t \setminus \gamma_t$ [12, 13]. Assuming that

$$b_{mkl} := h_{mi}b^i_{kl}, \qquad \gamma_{mkl} := h_{mi}\gamma^i_{kl}, \tag{6}$$

it can easily be shown (see [12]) that

$$\gamma_{mkl} = (\partial_k h_{ml} + \partial_l h_{mk} - \partial_m h_{kl})/2 + (b_{klm} + b_{mkl} - b_{lmk})/2. \tag{7}$$

We consider the system of equations defined by the last relation in (4). This is an overdetermined system of first-order partial equations. Commutation of the coordinate fields $(\partial_r, \partial_\varphi, \partial_s)$ in $U_t \setminus \gamma_t$ is a condition of solvability of this system. Applying this condition to the system, we have

$$R_{gh,ij} := \partial_i\gamma_{ghj} - \partial_j\gamma_{ghi} + h^{kl}\gamma_{kgj}\gamma_{lhi} - h^{kl}\gamma_{kgi}\gamma_{lhj} + \gamma_{ghk}b^k_{ij} = 0, \tag{8}$$

where h^{kl} are the elements of the inverse matrix of h_{kl}. Equation (8) is called the condition of zero curvature of the connectivity of γ^m_{kl}. Relations (5)–(8) represent the necessary and sufficient conditions for the fact that the functions h_{kl} determine the coordinate system (r, φ, s).

We write the condition under which the coordinate φ is the fluid velocity-field potential in terms of metric components. As the fluid is incompressible, this condition implies that φ is a harmonic (in $U_t \setminus \gamma_t$) function, i.e., $\Delta\varphi = 0$, where Δ is a three-dimensional Laplace operator. Hence, in the basis $(\partial_1, \partial_2, \partial_3)$, we have

$$r^{-1}h^{12} + h^{kl}\gamma^2_{kl} = 0. \tag{9}$$

Let us formulate the problem. Specifying the above requirements for the coordinate system (r, φ, s), we determine the "extended" (with respect to r) variable $\rho := \varepsilon^{-1}r$ and require that the following asymptotic expansion holds:

$$\boldsymbol{x} = \boldsymbol{x}_0 + \varepsilon\boldsymbol{x}_1 + \ldots + \varepsilon^n\boldsymbol{x}_n + \ldots\ . \tag{10}$$

Then,

$$\boldsymbol{e}_k = \boldsymbol{e}_{k;0} + \varepsilon\boldsymbol{e}_{k;1} + \ldots + \varepsilon^n\boldsymbol{e}_{k;n} + \ldots\ , \qquad h_{kl} = h_{kl;0} + \varepsilon h_{kl;1} + \ldots + \varepsilon^n h_{kl;n} + \ldots\ . \tag{11}$$

We also require satisfaction of the boundary condition

$$\lim_{\rho\to 0} h_{kl;n} = 0 \qquad (k = l,\ n \geqslant 1) \tag{12}$$

and the relations

$$\boldsymbol{x}_0 := \boldsymbol{X}(t,s), \qquad h_{kl;0} := \delta_{kl}, \tag{13}$$

where δ_{kl} is the Kronecker symbol.

In the present paper, we confine ourselves to calculating the first approximation to h_{kl}. As the governing system, we take Eqs. (4)–(7), the zero-curvature condition (8), and the harmonicity condition (9).

3. Calculation of the Metrics Components in the Coordinates $\boldsymbol{r}$, $\boldsymbol{\varphi}$, and $\boldsymbol{s}$. In the calculations, we use a set of simple equalities $h^{kl}_{;0} = \delta^{kl}$, $h^{kl}_{;1} = -h_{kl;1}$, $b_{klm;0} = b^{k}_{lm;0}$, etc., derived from formulas of the form of (6). After substitution of the corresponding asymptotic relations into these formulas and separation of terms with appropriate exponents ε, these formulas relate geometrical quantities to corresponding metrically dual quantities.

We write the system of equations for determining the quantities $h_{kl;1}$. Using (7), we express conditions (8) and (9) in terms of metric components, replace the metric components by their asymptotic expressions, and equate to zero the sums of terms with ε^{-1} and ε^0, respectively. As a result, considering the components $R_{12,12}$, $R_{13,13}$, $R_{23,23}$, $R_{12,13}$, and $R_{12,23}$, from (8), we obtain the equations

$$-\frac{1}{2}\frac{\partial^2 h_{22;1}}{\partial\rho^2} - \frac{1}{2\rho^2}\frac{\partial^2 h_{11;1}}{\partial\varphi^2} - \frac{1}{\rho}\frac{\partial h_{22;1}}{\partial\rho} + \frac{1}{2\rho}\frac{\partial h_{11;1}}{\partial\rho} + \frac{1}{\rho}\frac{\partial^2 h_{12;1}}{\partial\varphi\,\partial\rho} + \frac{1}{\rho^2}\frac{\partial h_{12;1}}{\partial\varphi} = 0; \tag{14}$$

$$-\frac{1}{2}\frac{\partial^2 h_{33;1}}{\partial\rho^2} = 0, \qquad -\frac{1}{2\rho^2}\frac{\partial^2 h_{33;1}}{\partial\varphi^2} - \frac{1}{2\rho}\frac{\partial h_{33;1}}{\partial\rho} = 0; \tag{15}$$

$$\frac{\partial}{\partial\rho}\Big(\frac{1}{2\rho}\frac{\partial h_{13;1}}{\partial\varphi} - \frac{1}{2\rho}\frac{\partial\rho h_{23;1}}{\partial\rho}\Big) = 0, \quad \frac{\partial}{\partial\varphi}\Big(\frac{1}{2\rho}\frac{\partial h_{13;1}}{\partial\varphi} - \frac{1}{2\rho}\frac{\partial\rho h_{23;1}}{\partial\rho}\Big) = 0, \tag{16}$$

and from (9), we obtain the equation

$$-\frac{1}{2\rho}\frac{\partial}{\partial\varphi}\Big(h_{11;1} - h_{22;1} + h_{33;1}\Big) + \frac{\partial h_{12;1}}{\partial\rho} = 0. \tag{17}$$

Eliminating the unknown variable $h_{11;1}$ from (14) using (17), with accuracy to an arbitrary function that does not depend on φ, we obtain

$$\Delta\Big(h_{22;1} - 2\int h_{12;1}\,d\varphi\Big) = -\frac{1}{\rho}\frac{\partial h_{33;1}}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 h_{33;1}}{\partial\varphi^2}, \tag{18}$$

where Δ is a two-dimensional Laplace operator:

$$\Delta := \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}.$$

Equations (15)–(18) are the desired equations.

The solution of system (15) satisfying boundary condition (12) has the form

$$h_{33;1} = -\psi^*\rho e^{i\varphi} + \text{c.c.}, \tag{19}$$

where $\psi(t,s)$ is the constant of integration. With allowance for (12) and (19) and without seeking generality, we take the following equation as a solution of Eq. (18):

$$h_{22;1} - 2\int h_{12;1}\,d\varphi = \ln(\alpha\rho)\psi^*\rho e^{i\varphi} + \text{c.c.}, \tag{20}$$

where $\alpha(t,s)$ is an arbitrary positive function. As solutions of system (16), we consider the arbitrary functions $h_{13;1}$ and $h_{23;1}$ that satisfy the equation

$$\frac{\partial h_{13;1}}{\partial\varphi} - \frac{\partial\rho h_{23;1}}{\partial\rho} = 0.$$

The coefficient $h_{11;1}$ is obtained by substitution of (19) and (20) into (17). The choice of $h_{12;1}$ is arbitrary.

We determine the geometrical meaning of the constant of integration ψ. Into the last equation of (4), we substitute expansion (11) and the known expansion of connectivity coefficients determined with accuracy to ε^0. In particular, for ε^{-1} and ε^0, we have

$$\frac{\partial \boldsymbol{e}_{1;0}}{\partial \varphi} = \boldsymbol{e}_{2;0}, \qquad \frac{\partial \boldsymbol{e}_{2;0}}{\partial \varphi} = -\boldsymbol{e}_{1;0}, \qquad \frac{\partial \boldsymbol{e}_{k;0}}{\partial \rho} = \frac{\partial \boldsymbol{e}_{3;0}}{\partial \varphi} = 0; \tag{21}$$

$$\frac{\partial \boldsymbol{e}_{1;0}}{\partial s} = -\frac{1}{2}\,\psi^* \mathrm{e}^{i\varphi} \boldsymbol{e}_{3;0} + \text{c.c.}, \qquad \frac{\partial \boldsymbol{e}_{3;0}}{\partial s} = \frac{1}{2}\,\psi^* \mathrm{e}^{i\varphi} (\boldsymbol{e}_{1;0} + i\boldsymbol{e}_{2;0}) + \text{c.c.} \tag{22}$$

The solution of system (21) is written as

$$\boldsymbol{e}_{1;0} = \boldsymbol{N}^* \mathrm{e}^{i\varphi}/2 + \text{c.c.}, \qquad \boldsymbol{e}_{2;0} = \boldsymbol{N}^* i\mathrm{e}^{i\varphi}/2 + \text{c.c.}, \qquad \boldsymbol{e}_{3;0} = \boldsymbol{t}. \tag{23}$$

As follows from (13), a pair of $\langle \boldsymbol{N}, \boldsymbol{t} \rangle$ forms a normalized dyad of vectors, and the vector $\boldsymbol{t}$ is a tangent to γ_t. Substitution of (23) into (22) yields a system of equations whose form coincides with that of the Serret–Frenet system (2). Hence, ψ is the natural curvature and $\langle \boldsymbol{N}, \boldsymbol{t} \rangle$ is the natural reference of the curve γ_t.

Thus, the problem of determining the external field (in the sense indicated above) is solved.

Remark 1. Reverting to the initial variable r, we find that the field h_{kl} depends on the parameter α/ε, which can take arbitrary positive values. Varying this parameter, we obtain a set of external fields. The question of whether this set contains a filament-induced field remains open.

4. Hydrodynamic Equations in the Coordinates r, φ, and s. Below, we relate the fluid flow with moving curvilinear coordinates r, φ, and s in $U_t \setminus \gamma_t$.

We use u^k, v^k, and h^k to denote the velocity-field components in the basis $(\partial_1, \partial_2, \partial_3)$ at time t for absolute fluid motion, relative fluid motion, and the motion of the coordinate system, respectively. Let $(\theta_1, \theta_2, \theta_3)$ denotes the basis in $U_t \setminus \gamma_t$ which is dual to the basis $(\partial_1, \partial_2, \partial_3)$. We set $u_k := h_{kl} u^l$, $v_k := h_{kl} v^l$, and $h_k := h_{kl} h^l$. The quantities u_k, v_k, and h_k are the components in the basis $(\theta_1, \theta_2, \theta_3)$ of the fields that are metrically dual to the indicated velocity fields. Below, we use only the quantities u_k, v_k, and h_k (and not the quantities u^k, v^k, and h^k); therefore, the term "metrically dual" in the names of the corresponding fields is omitted. Thus, the obvious relation $u_k = v_k + h_k$ should be interpreted as the "equality of the velocity field of absolute fluid motion to the sum of the velocity fields of relative fluid motion and the motion of the coordinate system." We note that for the components h_k, an expression similar to that for the components h_{kl} in (4) can be written as

$$h_k = (\boldsymbol{e}_k, \boldsymbol{e}_0) \qquad \Big(\boldsymbol{e}_0 := \frac{\partial \boldsymbol{x}}{\partial t}\Big). \tag{24}$$

In the moving curvilinear coordinates r, φ, and s in the basis $(\theta_1, \theta_2, \theta_3)$ in $U_t \setminus \gamma_t$, the Euler equations and discontinuity equations have the form

$$\frac{\partial}{\partial t}\, u_i + v_m h^{ml} (\partial_l u_i - \partial_i u_l + u_k b^k_{li}) + \partial_i \Big(p + \frac{1}{2}\, h^{kl} u_k u_l\Big) = 0; \tag{25}$$

$$-h^{kl} \partial_k u_l + h^{kl} \gamma^m_{lk} u_m = 0. \tag{26}$$

The validity of this representation can be shown by converting from the basis $(\theta_1, \theta_2, \theta_3)$ of the coordinates considered to a Cartesian basis of natural inertial coordinates in $\mathbb{R}^3$.

5. Formulation of the Problem of Determining the Vortex Dynamics. We assume that the region U_t contains not only the vortex filament γ_t but the vortex studied. We take that circulations of both the filament and the vortex coincide. We take the boundary-surface equation for the vortex in the form $r = r_B(t, \varphi, s)$.

Let us formulate general requirements for the flow. We take ε as a small parameter of the problem, convert to the variable $\rho := \varepsilon^{-1} r$, and preserve all the previous assumptions on the asymptotic behavior of the metrics. We also require that

$$v_k = \varepsilon^{-1} v_{k;-1} + v_{k;0} + \ldots + \varepsilon^n v_{k;n} + \ldots, \qquad h_k = h_{k;0} + \varepsilon h_{k;1} + \ldots + \varepsilon^n h_{k;n} + \ldots,$$

$$\rho_B = \rho_0 + \varepsilon \rho_1 + \ldots + \varepsilon^n \rho_n + \ldots \qquad (\rho_B := \varepsilon^{-1} r_B),$$

and

$$\lim_{\rho \to 0} v_{k;n} = 0, \qquad \rho_0 = \rho_0(t, s) \qquad (k = 1, 2;\ n \geqslant -1). \tag{27}$$

For the external field, we take

$$v_{1;-1} = 0, \qquad v_{2;-1} = v(\rho), \qquad v_{3;-1} = w(\rho), \tag{28}$$

where v and w are arbitrary functions in U_t and $v \to 0$ as $\rho \to 0$. We assume that this field is continuous over the entire U_t.

The external field of absolute velocities is determined as a differential of the coordinate φ multiplied by $\Gamma/(2\pi)$ (Γ is the filament circulation). In the basis $(\theta_1, \theta_2, \theta_3)$, the field has the components

$$u_1 = u_3 = 0, \qquad u_2 = \Gamma/(2\pi r).$$

Hence,

$$u_{1;-1} = u_{3;-1} = 0, \qquad u_{2;-1} = \Gamma/(2\pi\rho), \qquad u_{k;n} = 0 \quad (n \geqslant 0). \tag{29}$$

We define the dynamics of the vortex from the joining condition as $\rho \to \infty$ for the internal and external fields of absolute velocities up to a zeroth approximation. The velocity field for the coordinate system motion is determined from Eq. (24), and the internal field of relative fluid velocities is determined from Eqs. (25) and (26).

6. The Equation of Vortex Dynamics. We apply the joining condition to the initial approximations of the internal and external fields of absolute fluid velocities. By virtue of the conditions of flow potentiality outside the vortex and the continuity conditions for the velocity fields, in passage through the vortex boundary, from Eqs. (28) and (29), we have $v = \Gamma/(2\pi\rho)$ and $w = 0$ for $\rho \geqslant \rho_0$, i.e., in the initial approximation, the internal and external velocity fields coincide outside the vortex and at the vortex boundary.

We calculate the zeroth approximation to the velocity field of the coordinate system. We substitute (10) and (11) into (24) and separate terms with ε^0 in the equation obtained. With allowance for (23), we have

$$h_{1;0} = (\boldsymbol{X}_t, \boldsymbol{N}^*)\mathrm{e}^{i\varphi}/2 + \text{c.c.}, \qquad h_{2;0} = (\boldsymbol{X}_t, \boldsymbol{N}^*)i\mathrm{e}^{i\varphi}/2 + \text{c.c.}, \qquad h_{3;0} = (\boldsymbol{X}_t, \boldsymbol{t}),$$

where $\boldsymbol{X}_t := \partial \boldsymbol{X}/\partial t$. We note that

$$\frac{\partial h_{1;0}}{\partial\varphi} = h_{2;0}, \qquad \frac{\partial h_{2;0}}{\partial\varphi} = -h_{1;0}. \tag{30}$$

Let us calculate the zeroth approximation to the radial and rotational components of the internal field of relative fluid velocities. With allowance for (9), we write (26) in the form

$$-h^{kl}\partial_k u_l + h^{kl}\gamma^1_{lk}u_1 - h^{12}u_2/r + h^{kl}\gamma^3_{lk}u_3 = 0. \tag{31}$$

We substitute the asymptotic expressions for h^{kl}, u_k, and γ^m_{kl} into Eq. (31) and separate terms with ε^{-1}. With allowance for (30), we have

$$-\frac{1}{\rho}\frac{\partial\rho v_{1;0}}{\partial\rho} - \frac{1}{\rho}\frac{\partial v_{2;0}}{\partial\varphi} + \omega h_{12;1} + \frac{\partial w h_{13;1}}{\partial\rho} + \frac{w}{\rho}\Big(h_{13;1} + \frac{\partial h_{23;1}}{\partial\varphi}\Big) = 0, \tag{32}$$

where $\omega := v_\rho + v/\rho$. Equation (32) is satisfied by introducing the stream function $\psi_{12;1}$:

$$v_{1;0} = \frac{1}{\rho}\frac{\partial\psi_{12;1}}{\partial\varphi} + w h_{13;1}, \qquad v_{2;0} = -\frac{\partial\psi_{12;1}}{\partial\rho} + \rho\omega\int h_{12;1}\,d\varphi + w h_{23;1}. \tag{33}$$

We exclude the pressure from the first two equations ($i = 1, 2$) of system (25) and substitute the asymptotic expressions for v_k, h_k, and h^{kl} with the coefficients $v_{1;0}$ and $v_{2;0}$ in the form of (33) into the obtained equation. For ε^{-2}, we have

$$\Delta\frac{\partial\psi_{12;1}}{\partial\varphi} - \frac{\omega_\rho}{v}\frac{\partial\psi_{12;1}}{\partial\varphi} + \omega\Big(\frac{\partial h_{22;1}}{\partial\varphi} - 2h_{12;1}\Big) + \frac{w w_\rho}{v}\frac{\partial h_{33;1}}{\partial\varphi} = 0, \tag{34}$$

where Δ is a two-dimensional Laplace operator. We will seek solutions of Eq. (34) in the form

$$\psi_{12;1} = b(t, \rho, s) + c(\rho)\psi^*\mathrm{e}^{i\varphi} + \text{c.c.} \tag{35}$$

Substitution of (19), (20), and (35) into (34) yields $c_{\rho\rho} + c_\rho/\rho - c/\rho^2 - \omega_\rho c/v = -\omega\rho\ln(\alpha\rho) + \rho w w_\rho/v$. Integration of the above equation with allowance for (27) gives

$$c = v\int_0^\rho \frac{d\eta}{\eta v^2(\eta)}\int_0^\eta (-\zeta^2 v\omega\ln(\alpha\zeta) + \zeta^2 w w_\zeta)\,d\zeta. \tag{36}$$

Equations (33), (35), and (36) define the zeroth approximation to the radial and rotational components of the internal fields of relative fluid velocities.

We join the zeroth approximations of the internal and external fields of absolute fluid velocities. In particular, for the internal field, we have

$$\lim_{\rho\to\infty}(v_{1;0}+h_{1;0})=0.$$

After substitution of the explicit expressions for $v_{1;0}$ and $h_{1;0}$ and several simple transformations, we obtain the desired equation of vortex dynamics

$$\frac{\partial \boldsymbol{X}}{\partial t}=\frac{\Gamma}{8\pi}\left(\ln(\alpha\rho_0)-A(\rho_0)\right)i\psi^*\boldsymbol{N}+\text{c.c.}+(\boldsymbol{X}_t,\boldsymbol{t})\boldsymbol{t}, \tag{37}$$

where

$$A(\rho):=\frac{4\pi^2}{\Gamma^2}\left(\int\limits_0^\rho \zeta v^2\,d\zeta-2\int\limits_0^\rho \zeta w^2\,d\zeta\right).$$

We note that the expression for $A(\rho_0)$ coincides with the corresponding expression obtained in [2–6]. The value of this parameter is determined by the initial approximation to the axial and azimuthal fluid velocities in the vortex core. For example, for the initial fluid velocity distribution, corresponding to the rigid-body rotation, we have $A(\rho_0)=1/4$.

Without loss for generality, the third term on the right side of Eq. (37) can be set equal to zero. The procedure for eliminating this term is as follows: starting from Sec. 4, it is necessary to convert from the coordinate system (r,φ,s) to the system (r,φ,ξ), where ξ is a parameter along the filament γ_t for which $(\partial\boldsymbol{X}(t,\xi)/\partial t,\boldsymbol{t})=0$. In this case, all quantities in the formulas should be still related to the basis $(\partial_1,\partial_2,\partial_3)$. Then, the calculations performed under Sec. 6 remain unchanged, except for formula (37), in which the third term is absent.

If the initial approximation to the vortex core radius is constant along the filament and with time, then according to Remark 1 in Sec. 3, the quantity $\ln(\alpha\rho_0)$ is a parameter of the external field. By construction, this parameter is an arbitrary function of t and s. The choice of this function specifies both the vortex dynamics and the flow in which this dynamics occurs. In particular, the integrated vortex dynamics occurs if this parameter is a constant.

Conclusions. A new method for determining the dynamics of a thin curved vortex in the potential flow of an ideal incompressible fluid was proposed. Traditionally, the vortex dynamics was determined under the assumption that the potential flow, at least at large distances from the vortex, satisfies the Biot–Savart law. Because the induced effect on the entire vortex flow was difficult to estimate, the concept of local induction was used. In the new method proposed here, the indicated assumption is ignored and the flow is constructed on the basis of the requirement that the external field of fluid velocities coincide locally with the field induced by an appropriate straight vortex.

The equation of vortex dynamics was obtained, which coincided in form with the classical equation of local induction. The parameters of the new equation have the meaning of flow parameters, and the coefficient of this equation does not necessarily exceeds unity.

Unlike the traditional methods, the new method for determining the vortex dynamics provides an analytical proof of the existence of flows that show this dynamics, particularly, the integrated dynamics of vortex.

The method for studying fluid flows by using coordinates related specifically to the potential flow component is of interest. In these coordinates, the potential part of flow is trivial, i.e., it is an *a priori* known constant. Hence, investigation of full flows reduces to investigation of the vortex components of these flows.

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